

Abstract Algebra I

Exercise Sheet 4: Cosets

Equivalence Relations

Question 1. For this question $\mathbb{N}$ denotes the natural numbers including zero. Decide whether the following are equivalence relations:

- for $a, b \in \mathbb{Z}$ let $a \sim b$ if $a = b + 3k$ for some $k \in \mathbb{Z}$;
- for $a, b \in \mathbb{Z}$ let $a \sim b$ if $a = b + 1 + 3k$ for some $k \in \mathbb{Z}$;
- for $a, b \in \mathbb{Z}$ let $a \sim b$ if $a = b + 2 + 3k$ for some $k \in \mathbb{Z}$;
- $\leq$ on $\mathbb{Z}$;
- for $a, b \in \mathbb{N}$ let $a \sim b$ if $\gcd(a, b) = 1$;
- for $a, b \in \mathbb{Z}$ let $a \sim b$ if $|a| = |b|$;
- for $a, b \in \mathbb{R}$, $a, b > 0$, let $a^2 \sim b^2$ if $a - b \in \mathbb{Z}$;

Cosets

Question 2. For the following groups and subgroups compute all the left and right cosets. Determine whether the subgroup is normal. Decide whether the groups and subgroups are simple (whether it has non trivial normal subgroups).

- $G = \mathbb{Z}_6$ and $H = \mathbb{Z}_3$.
- $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$.
- $G = \mathbb{Z}_9$ and $H = \langle 3 \rangle$.
- $G = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $H = \langle (0, 1) \rangle$.
- $G = \mathbb{Z} \times \mathbb{Z}$ and $H = \langle (1, 0), (0, 2) \rangle$.
- Let G be the group of symmetric transformations of a regular pentagon. Let H be the subgroup of rotations of the pentagon.
- Let G be the group of symmetric transformations of a regular pentagon. Let H be any subgroup generated by a single flip of the pentagon.

Question 3. Consider a regular tetrahedron with vertices $v_1, \dots, v_4$, with group of symmetric transformations S_4 and subgroup of symmetric rotations A_4 . Consider a fixed vertex, say v_1 and consider the subsets of S_4 and A_4 respectively that *fix* v_1 , that is to say, do not change the labelling of this vertex. These subsets are called the *stabilisers* of v_1 .

- Prove that the stabiliser of v_1 in S_4 is a subgroup of S_4 . Find its left and right cosets. Is it normal?
- Prove that the stabiliser of v_1 in A_4 is a subgroup of S_4 . Find its left and right cosets. Is it normal?
- Find a normal subgroup of A_4 .

Question 4. The 5-cell is the 4-dimensional simplex, the simplest shape one can construct of same length edges in 4-dimensions. Let G be the group of symmetry preserving rotations of the 5-cell (A_5). Let H be the stabiliser of a single vertex, $H < G$. List the cosets of H . Is H normal in G ?